

A Novel Time-Varying Coding Scheme for the Gaussian Broadcast Channel with Feedback

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Abstract—In this paper, we generalize a coding strategy for the Gaussian broadcast channel with feedback, analyze error probabilities and achievable rate region for this coding strategy by using the iterated random function theory [1], [2], [6]. By changing some parameters in our scheme, we realize a variant of the Ozarow-Leung's code for the general two-user broadcast channel with feedback. A capacity-achieving coding scheme for the degraded broadcast channel with feedback is drawn. More interestingly, we come to a coding scheme for the symmetric Gaussian broadcast channel with feedback which achieves the same sum-rate as the LQG code [18] and strictly outperforms the Kramer code [14]. The fact that our coding scheme is a variant of the Kramer's code or time-varying posterior matching code [1] may give useful information to solve some open problems proposed in [19], [20].

Index Terms—Gaussian Broadcast Channel with Feedback, Feedback, Posterior Matching, Iterated Function Systems.

I. INTRODUCTION

The capacity of the broadcast channel is a well-known open problem. However, it is known that feedback can increase the capacity of this kind of channel. Specially, Ozarow and Leung [4] proved that feedback could increase the capacity of the two-user additive white Gaussian broadcast channel (AWGN-BC) with feedback by providing a means of cooperation between the receivers and the sender. Kramer [14] extended this encoding-decoding scheme for more than two users. Later, Elia [7] showed that the achievable rate region obtained by Ozarow and Leung [4] could be enlarged by using robust control theory. Ardestanizadeh et al. [18] proposed an LQG (Linear Quadratic Gaussian) based encoding-decoding scheme for the symmetric Gaussian broadcast channels with feedback and shown that their LQG code to provide the same achievable rate as Elia [7] for the symmetric two-user Gaussian broadcast channel with feedback and strictly outperform Kramer code [14] for the symmetric Gaussian broadcast channel with feedback when the number of users is greater than 3. The LQG code is derived based on a mapping from a feedback control problem to a linear code for the AWGN-BC with feedback. The set of achievable rates is determined by the eigenvalues of the open-loop matrix of a linear system and the power constraint is related to the minimum power needed to stabilize the system using a feedback control signal.

In a more general direction, Gaspar et al. [19], [20] proposed a coding scheme for the Gaussian broadcast channel

with correlated noises for the two user case with arbitrary noise covariance and for the more user case when noise samples at receivers are all multiples of each other. For example, they showed that for all noise correlation other than ± 1 , the gap between the sum-rate of their scheme achieves and the full-cooperation bound vanishes as the signal-to-noise ratio tends to infinity. Although their coding scheme works well in the asymptotic regime, it does not work well when the input power is not sufficiently large.

In this paper, using analysis tool, we firstly analyze the achievable rate region for a general coding strategy for the Gaussian broadcast channel with feedback that includes all the coding schemes [1], [4], [14]. Then, we prove that a variant of the Ozarow-Leung [4] scheme can be obtained from this strategy with the same achievable rate region. Moreover, we come to an coding scheme for the multi-user degraded AWGN-BC with feedback. Furthermore, we proposed a coding scheme for the non-degraded AWGN-BC with feedback to achieve the same sum-rate as the LQG code. Our encoding scheme is a variant of the Kramer's code, hence it shows a potential of this method to design a coding scheme which achieves not only the asymptotic capacity [19], [20], but also provides good performance in non-asymptotic settings.

The rest of this paper is organized as follows. Section II presents the channel model and some mathematical preliminaries. Section III proposes a general time-varying coding scheme for Gaussian broadcast channel with feedback. Section IV analyses the achievable rate region and error probabilities for this general scheme. Section V shows that Ozarow-Leung coding scheme is a variant of this coding scheme. Section VI proposed a coding scheme for the degraded Gaussian broadcast channel with feedback. Section VII designs a coding scheme achieving the same sum-rate as the LQG code. Finally, Section VIII concludes this paper.

II. CHANNEL MODEL AND PRELIMINARIES

A. Mathematical notations [1]

Upper-case letters, their realizations by corresponding lower-case letters, denote random variables. A real-valued random variable X is associated with a distribution $\mathbb{P}_X(\cdot)$ defined on the usual Borel σ -algebra over \mathbb{R} , and we write $X \sim \mathbb{P}_X$. The cumulative distribution function (c.d.f.) of

X is given by $F_X(x) = \mathbb{P}_X((-\infty, x])$, and their inverse c.d.f is defined to be $F_X^{-1}(t) := \inf\{x : F_X(x) > t\}$. The uniform probability distribution over $(0, 1)$ is denoted through \mathcal{U} . The composition function $(f \circ g)(x) = f(g(x))$. In addition, $\mathbf{Y}_p^{(q,m)} := (Y_p^{(m)}, Y_{(p+1)}^{(m)}, \dots, Y_q^{(m)})$ for $p \leq q$, $\text{tr}(\mathbf{A})$ is defined as the trace of the matrix \mathbf{A} . In this paper, we use the following lemma:

Lemma 1: Let X be a continuous random variable with $X \sim \mathbb{P}_X$ and Θ be an uniform distribution random variable, i.e. $\Theta \sim \mathcal{U}$ be statistical independent. Then $F_X^{-1}(\Theta) \sim \mathbb{P}_X$ and $F_X(X) \sim \mathcal{U}$.

Proof: Refer to [2] for the proof. ■

Big O notation (with a capital letter O, not a zero), also called Landau's symbol, is a symbolism used in complexity theory, computer science, and mathematics to describe the asymptotic behavior of functions. Basically, it tells you how fast a function grows or declines. For the formal definition, suppose $f(n)$ and $g(n)$ are two functions defined on positive integer number. We write

$$f(n) = O(g(n))$$

(or $f(n) = O(g(n))$ for $n \rightarrow \infty$ to be more precise) if and only if there exists constants N and C such that

$$|f(n)| \leq C|g(n)| \quad \text{for all } n > N$$

Intuitively, this means that f does not grow faster than g .

In addition to big O notations, another Landau symbol is used in mathematics: the little o. Formally, we write $f(n) = o(g(n))$ for $n \rightarrow \infty$ if and only if for every $C > 0$ there exists a real number N such that for all $n > N$ we have $|f(n)| < C|g(n)|$. If $g(n) \neq 0$, this is equivalent to $\lim_{n \rightarrow \infty} f(n)/g(n) = 0$.

A Hadamard matrix [15] of order n is an $(n \times n)$ matrix of +1s and -1s such that $\mathbf{H}\mathbf{H}^T = n\mathbf{I}$. In fact, it is not yet known for which values of n an \mathbf{H}_n does exists. However, we know that if a Hadamard matrix of order n exists, then n is 1, 2, 4, or a multiple of 4. Moreover, if n is of the form 2^m , m a positive integer, we can construct \mathbf{H}_n by using the Sylvester method. Besides, the Paley construction, which uses quadratic residues, can be used to construct Hadamard matrices of order n , where n is of the form $(p+1)$, p is a prime, and n is a multiple of 4.

B. Gaussian Broadcast Channel with Feedback

Consider the communication problem between one sender and M receivers over a broadcast channel with additive Gaussian noise (AWGN-BC) when channel outputs are noiselessly fed back to all the senders (Figure 1). Let Θ_m be a random message point uniformly distributed over the unit interval that will intend to transmit from the transmitter to the receiver $m \in \{1, 2, \dots, M\}$. At each time n , the received signal at the receiver m is

$$Y_n^{(m)} = X_n + Z_n + Z_n^{(m)}$$

where $X_n \in \mathbb{R}$ is the transmitted symbol by the sender at time n ; $Y_n^{(m)} \in \mathbb{R}$ is the received signal by the receiver m at

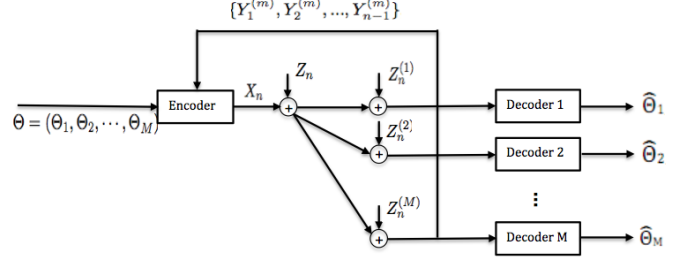


Fig. 1. M-user Gaussian Broadcast Channel with Feedback

the time n . Besides, $\{Z_n\}$ is a common white Gaussian noise component with variance σ^2 , and $Z_n^{(m)}$ are separate white noise components with variance σ_m^2 . For the non-physically degraded white Gaussian broadcast channel, we can set $\sigma^2 = 0$. Also, we assume that output symbols are casually fed back to the sender and that the transmitted symbol X_n at time n can depend on both the message $\Theta_1, \Theta_2, \dots, \Theta_M$ and the previous channel output sequences $\mathbf{Y}^{(n-1,m)} := (Y_1^{(m)}, Y_2^{(m)}, \dots, Y_{n-1}^{(m)})$, $\forall m \in \{1, 2, \dots, M\}$.

A *transmission scheme* for a Gaussian broadcast channel is a measurable transmission function $g_n : (0, 1) \times (0, 1) \times \dots \times \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \dots \times \mathbb{R}^{n-1}$, so that the input to the channel generated by the transmitter is given by

$$X_n = g_n(\Theta_1, \dots, \Theta_M, \mathbf{Y}^{(n-1,1)}, \mathbf{Y}^{(n-1,2)}, \dots, \mathbf{Y}^{(n-1,M)})$$

A *decoding rule* for a Gaussian broadcast channel are sequences of measurable mappings $\{\Delta_n^{(m)} : \mathbb{R}^n \Rightarrow \mathcal{E}\}_{n=1}^\infty$, where \mathcal{E} is the set of all open intervals in $(0, 1)$. We refer $\Delta_n^{(m)}(y^{(n,m)})$ as to the decoded interval at the receiver m . The error probabilities at time n associated with a transmission scheme and a decoding rule, is defined as

$$p_n^{(m)}(e) := \mathbb{P}(\Theta_m \notin \Delta_n^{(m)}(\mathbf{Y}^{(n,m)}), \quad \forall m \in \{1, 2, \dots, M\}$$

and the corresponding achievable rate region at time n is defined to be

$$\left\{ (R_n^{(1)}, \dots, R_n^{(M)}) : R_n^{(m)} := -\frac{1}{n} \log |\Delta_n^{(m)}(\mathbf{Y}^{(n,m)})| \right\}$$

We say that a transmission scheme together with a decoding rule achieve a rate region (R_1, R_2, \dots, R_M) over a Gaussian broadcast channel if for $m \in \{1, 2, \dots, M\}$ we have

$$\lim_{n \rightarrow \infty} \mathbb{P}(R_n^{(m)} < R_m) = 0, \quad \lim_{n \rightarrow \infty} p_n^{(m)}(e) = 0 \quad (1)$$

The rate vector is achieved within input power constraint P , if in addition

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n E(X_k)^2 \leq P \quad (2)$$

An *optimal fixed rate* decoding rule for a Gaussian broadcast channel with feedback with rate region (R_1, R_2, \dots, R_M) is one that decodes a vector of fixed length intervals

$\{(J_1, J_2, \dots, J_M) : |J_m| = 2^{-nR_m}, \forall m\}$, whose marginal posteriori probabilities are maximal, i.e.,

$$\Delta_n^{(m)}(y^{(n,m)}) = \operatorname{argmax}_{J_m \in \mathcal{E}: |J_m| = 2^{-nR_m}} \mathbb{P}_{\Theta_m|Y^n}(J_m|y^{(n,m)})$$

An *optimal variable rate* decoding rule with target error probabilities $p_e^{(m)}(n) = \delta_n^{(m)}$ is one that decodes a vector of minimal-length intervals (J_1, J_2, \dots, J_M) with accumulated marginal posteriori probabilities exceeds corresponding targets, i.e.,

$$\Delta_n^{(m)}(y^{(n,m)}) = \operatorname{argmin}_{J_m \in \mathcal{E}: \mathbb{P}_{\Theta_m|Y^n}(J_m|y^{(n,m)}) \geq 1 - \delta_n^{(m)}} |J_m|$$

Both decoding rules make use of the marginal posterior distribution of the message point $\mathbb{P}_{\Theta_m|Y^n}$ which can calculate online at the transmitter and the receivers. Refer [1], [2] for more details.

Lemma II: The achievability in the definition (1) and (2) implies the achievability in the standard framework.

Proof: See the detailed proof in papers [1], [2], [18]. The idea is that if we transmit a message point which is uniformly distributed on (0,1) (no prior knowledge about Θ_m), we can find a decoded interval with length $2^{-nR_m^*}$, where R_m^* depends on the signal to noise ratio, such that the transmitted message point belongs to this interval in probability by repeating the decoding process infinitely many times ($n \rightarrow \infty$). Hence, at each transmission time slot n , we can choose M message points in (0,1) separated by $2^{-nR_m^*}$ with the property that their decoded intervals to be non-overlapped. Mapping message points $\{1, 2, \dots, 2^{nR_m^*}\}$ defined in the standard framework to those message points will complete our encoding in the traditional way. The error probabilities of the associated scheme can be shown to decay as the root square of the $p_n^{(m)}(e)$. ■

III. A GENERAL TIME-VARYING CODING SCHEME FOR GAUSSIAN BROADCAST CHANNEL WITH FEEDBACK

In this section, we proposed a novel time-varying encoding scheme and a variable-rate decoding strategy for the Gaussian broadcast channel with feedback. We refer our below encoding-decoding to as a *time-varying coding and decoding strategy* for the Gaussian broadcast channel with feedback. Assuming that the transmitter is trying to reliably send M different messages $\{\Theta_m\}_{m=1}^M$ to M receivers, where Θ_m is the intended message for the user m , $\Theta_m \sim \mathcal{U}$. Our general encoding-decoding scheme as following:

A. Encoding Scheme

1) At each time slot n for $n \geq 1$

- The transmitter broadcasts the following signal to all the receivers:

$$X_n = \beta_n \sum_{m=1}^M \alpha_n^{(m)} S_n^{(m)}$$

Here, β_n is a coefficient to be chosen to satisfy the input power constraint (2), and

$$\alpha_n = [\alpha_n^{(1)} \quad \alpha_n^{(2)} \quad \dots \quad \alpha_n^{(M)}]^T$$

is a modulated vector.

- The receiver m receives the signal

$$Y_n^{(m)} = \beta_n \sum_{m=1}^M \alpha_n^{(m)} S_n^{(m)} + Z_n + Z_n^{(m)}$$

- The receiver m feedbacks the random variable $Y_n^{(m)}$ to the transmitter.
- The transmitter creates M random variables $S_{n+1}^{(m)}$, $1 \leq m \leq M$ as the following:

$$S_{n+1}^{(m)} = \frac{1}{a_n^{(m)}} (S_n^{(m)} - b_n^{(m)} Y_n^{(m)})$$

Here, $a_n^{(m)}, b_n^{(m)}$ ($a_n^{(m)} > 0$) are some real number sequences depending on the network situations. In addition, the initial sequence $S_1^{(m)}$, $1 \leq m \leq M$ is defined as

$$S_1^{(m)} = F_S^{-1}(\Theta_m)$$

where S is a Gaussian random variable and $S \sim \mathcal{N}(0, P_0)$ for some value of power $P_0 > 0$.

B. Decoding Scheme

1) At the time slot n for $n \geq 1$:

- At the time slot n , the receiver m receives the signal

$$Y_n^{(m)} = \beta_n \sum_{m=1}^M \alpha_n^{(m)} S_n^{(m)} + Z_n + Z_n^{(m)}$$

- The receiver m selects a fixed decoded interval $J_1^{(m)} = (s_m, t_m) \subset \mathbb{R}$ with respect to $S_{n+1}^{(m)}$.
- Then, the receiver m estimates the decoded interval for the $S_1^{(m)}$

$$J_n^{(m)} = (T_n^{(m)}(s_m), T_n^{(m)}(t_m))$$

where

$$T_n^{(m)}(x) := w_1^{(m)} \circ w_2^{(m)} \dots \circ w_n^{(m)}(x)$$

and

$$w_n^{(m)}(x) := a_n^{(m)} x + b_n^{(m)} Y_n^{(m)}$$

(The positivity of the sequences $a_n^{(m)}$ ensures that the random functions $w_n^{(m)}(x)$ and $T_n^{(m)}(x)$ are monotone increasing in the variable x for any realization $y^{(n,m)}$ of the random vector $\mathbf{Y}^{(n,m)}$).

- Finally, the decoded interval for Θ_m is set to be

$$\Delta_n^{(m)}(\mathbf{Y}^{(n,m)}) := F_S(J_n^{(m)})$$

where $S \sim \mathcal{N}(0, P_0)$ and

$$F_S((a, b)) := \left(\int_{-\infty}^a f_S(x) dx \quad \int_{-\infty}^b f_S(x) dx \right)$$

($f_S(t)$ is the p.d. f. of the random variable S). Note that the upper and lower limits of the decoded interval are values of the c.d.f at points a, b .

IV. ERROR ANALYSIS FOR THE TIME-VARYING CODING SCHEME FOR THE GAUSSIAN BROADCAST CHANNEL WITH FEEDBACK

In this section, we evaluate the performance of the general time-varying posterior matching scheme in Section III.

Theorem I: Consider a real Gaussian broadcast channel with feedback having one transmitter and M receivers. Assuming that the *time-varying coding and decoding strategy* in section III is being employed for this channel. Under the condition that $0 < \limsup_{n \rightarrow \infty} a_n^{(m)} < 1$ and that $W_n := \sup_n E[S_n^{(m)}]^2$ is upper bounded, define

$$R_m^* := -\limsup_{n \rightarrow \infty} \log a_n^{(m)}$$

Then, without input power constraint consideration, the rate region $\{(R_1, R_2, \dots, R_M) : R_m < R_m^*\}$ is achievable and the error probabilities $p_n^{(m)}(\epsilon)$ decays to zero as following:

$$-\log p_n^{(m)}(\epsilon) = o\left(2^{2n(R_m^* - R_m)}\right), \quad \forall m \in \{1, 2, \dots, M\}$$

Proof: Denote $R_n^{(m)}$ as the instant rate to transmit the intended messages Θ_m to the receiver m . For any fixed rate R_m , we have

$$\begin{aligned} \mathbb{P}\left(R_n^{(m)} < R_m\right) &\stackrel{(a)}{=} \mathbb{P}\left(-\frac{1}{n} \log \left|\Delta_n^{(m)}(\mathbf{Y}^{(n,m)})\right| < R_m\right) \\ &= \mathbb{P}\left(\left|\Delta_n^{(m)}(\mathbf{Y}^{(n,m)})\right| > 2^{-nR_m}\right) \\ &\leq \mathbb{P}\left(\left|J_n^{(m)}\right| > 2^{-nR_m}/K_s\right) \end{aligned} \quad (3)$$

where

$$K_s = \sup_{x \in \mathbb{R}} \{f_S(x)\}$$

Here, (a) follows from the definition of the corresponding instant achievable rate vector at time n in the Section II.

Observe that by our definition, then for all $t, s \in \mathbb{R}$, we have

$$|w_n^{(m)}(t) - w_n^{(m)}(s)| = a_n^{(m)}|t - s| \quad (4)$$

Let $0 < a_m := \limsup_{n \rightarrow \infty} a_n^{(m)} < 1$, it is easy to see that $R_m^* = \log r_m^{-1} > 0$. For any rate $R_m < R_m^*$, we can find an $\epsilon > 0$ such that $R_m < \log(a_m + \epsilon)^{-1} < R_m^*$. Furthermore, since $a_m = \limsup_{n \rightarrow \infty} a_n^{(m)} < 1$, there exists an $N_\epsilon \in \mathbb{N}$ such that $\sup_{n \geq N_\epsilon} a_n^{(m)} < a_m + \epsilon$. Define $v_m := \sup_{1 \leq k \leq N_\epsilon} a_n^{(m)}$. From (3) and (4), we have

$$\begin{aligned} \mathbb{P}\left(R_n^{(m)} < R_m\right) &\leq \mathbb{P}\left(\left|J_n^{(m)}\right| > 2^{-nR_m}/K_s\right) \\ &\stackrel{(a)}{\leq} K_s 2^{nR_m} E\left[E\left(\left|w_1^{(m)} \circ w_2^{(m)} \dots \circ w_n^{(m)}(t_m)\right.\right.\right. \\ &\quad \left.\left.\left.- w_1^{(m)} \circ w_2^{(m)} \dots \circ w_n^{(m)}(s_m)\right|\right|\mathbf{Y}_2^{(n,m)}\right)\right] \\ &\stackrel{(b)}{\leq} K_s 2^{nR_m} v_m E\left[E\left(\left|w_2^{(m)} \circ w_3^{(m)} \dots \circ w_n^{(m)}(t_m)\right.\right.\right. \\ &\quad \left.\left.\left.- w_2^{(m)} \circ w_3^{(m)} \dots \circ w_n^{(m)}(s_m)\right|\right)\right] \end{aligned}$$

$$\begin{aligned} &\stackrel{(c)}{\leq} K_s 2^{nR_m} v_m^{N_\epsilon} E\left[E\left(\left|w_{N_\epsilon}^{(m)} \circ w_2^{(m)} \dots \circ w_n^{(m)}(t_m)\right.\right.\right. \\ &\quad \left.\left.\left.- w_{N_\epsilon}^{(m,k)} \circ w_2^{(m)} \dots \circ w_n^{(m)}(s_m)\right|\right)\right] \\ &\dots \stackrel{(d)}{\leq} K_s 2^{nR_m} v_m^{N_\epsilon} (a_m + \epsilon)^{(n-N_\epsilon)} |J_1^{(m)}| \end{aligned} \quad (5)$$

where (a) follows from the Markov's inequality and the law of iterated expectations, (b) follows from $v_m := \sup_{1 \leq n \leq N_\epsilon} a_n^{(m)}$, and (c) is a recursive application of the preceding transitions, (d) follows from $\sup_{n \leq N_\epsilon} a_n^{(m)} < a_m + \epsilon$ above and the recursive applications of the preceding transitions.

From (5), it is easy to see that a sufficient condition for $\mathbb{P}(R_n^{(m)} < R_m) \rightarrow 0$ is given by choosing $|J_1^{(m)}| = o\left(2^{n(\log(a_m + \epsilon)^{-1} - R_m)}\right)$.

Denote $Q(x)$ as the well-known tail function of the standard normal distribution and use the Chernoff bound of this function, we obtain

$$\begin{aligned} p_n^{(m)}(\epsilon) &= \mathbb{P}\left(\Theta_m \notin \Delta_n^{(m)}\left(\mathbf{Y}^{(n,m)}\right)\right) \\ &= \mathbb{P}\left(\Theta_m \notin F_S(J_n^{(m)})\right) \\ &= \mathbb{P}\left(S_1^{(m)} \notin J_n^{(m)}\right) = \mathbb{P}\left(S_n^{(m)} \notin J_1^{(m)}\right) \\ &\stackrel{(a)}{=} 2Q\left(\frac{|J_1^{(m)}|}{2}\right) \stackrel{(b)}{=} O\left(\exp\left(-\frac{|J_1^{(m)}|^2}{8W_n^{(m)}}\right)\right) \end{aligned} \quad (6)$$

Here, (a) follows from the fact that $J_1^{(m)}$ is symmetric if we set $s_m = -t_m$, and (b) follows from the Chernoff bound for the Q-function $0 < Q(x) \leq (1/2) \exp(-x^2/2), \forall x > 0$.

By the fact that $R_m < \log(a_m + \epsilon)^{-1} < R_m^*$, we have $|J_1^{(m)}| \rightarrow \infty$. Furthermore, with the assumption that W_n is upper bounded by some W , we have

$$\frac{|J_1^{(m)}|^2}{8W_n^{(m)}} \geq \frac{|J_1^{(m)}|^2}{8W_n} \geq \frac{|J_1^{(m)}|^2}{8W} \rightarrow \infty$$

To put it simply, if $R_m < R_m^*$, the error probabilities tend to zero as

$$-\log p_n^{(m)}(\epsilon) = o\left(2^{2n(\log(a_m + \epsilon)^{-1} - R_m)}\right) \approx o\left(2^{2n(R_m^* - R_m)}\right)$$

Remark: Since we can estimate R_m^* and know our desired rate R_m in advance, it is possible to choose ϵ by target. This means that the decoding algorithm is technically realizable. However, there is a tradeoff between the transmission rate R_m (the possible values of ϵ) and the code length n . If we transmit at the rate R_m very close to R_m^* , we need to choose ϵ to be very small. As a result, the required N_ϵ may be very large. Furthermore, the fact that $\log(a_m + \epsilon)^{-1}$ is very close to R_m also makes the error probabilities be slowly decayed to zero. To put it simply, the code length n may be very large if we transmit at the rate R_m is nearly R_m^* . On the contrary, being able to choose quite large ϵ makes the required N_ϵ smaller and the decay of error probabilities faster. ■

V. A VARIANT OF OZAROW-LEUNG'S CODING SCHEME
FOR THE GENERAL 2-USER GAUSSIAN BROADCAST
CHANNEL WITH FEEDBACK

Denote

$$\rho_n := \frac{E[S_n^{(1)} S_n^{(2)}]}{P/2}$$

In this case, we set

$$\begin{aligned} P_0 &= P/2 \\ \alpha_n^{(1)} &= 1 \\ \alpha_n^{(2)} &= g \operatorname{sgn}(\rho_n) \end{aligned}$$

Here, $\operatorname{sgn}(x) := 1$ if $x \geq 0$ and $\operatorname{sgn}(x) := -1$ if $x < 0$. We also define

$$\begin{aligned} \beta_n &= \sqrt{\frac{2}{1 + g^2 + 2g|\rho_n|}} \\ a_n^{(1)} &= \sqrt{\frac{\operatorname{var}(S_n^{(1)} | Y_n^{(1)})}{P/2}} \\ a_n^{(2)} &= \sqrt{\frac{\operatorname{var}(S_n^{(2)} | Y_n^{(2)})}{P/2}} \\ b_n^{(1)} &= \frac{E[S_n^{(1)} Y_n^{(1)}]}{\operatorname{var}(Y_n^{(1)})} \\ b_n^{(2)} &= \frac{E[S_n^{(2)} Y_n^{(2)}]}{\operatorname{var}(Y_n^{(2)})} \end{aligned}$$

By our transmission strategy, we have

$$X_n = \beta_n [S_n^{(1)} \alpha_n^{(1)} + S_n^{(2)} \alpha_n^{(2)}] = \beta_n [S_n^{(1)} + g \operatorname{sgn}(\rho_n) S_n^{(2)}]$$

Therefore, we have the output sequences at the two receivers are

$$\begin{aligned} Y_n^{(1)} &= \beta_n [S_n^{(1)} + g \operatorname{sgn}(\rho_n) S_n^{(2)}] + Z_n + Z_n^{(1)} \\ Y_n^{(2)} &= \beta_n [S_n^{(1)} + g \operatorname{sgn}(\rho_n) S_n^{(2)}] + Z_n + Z_n^{(2)} \end{aligned}$$

It follows that

$$\begin{aligned} E(X_n) &= E(Y_n^{(1)}) = E(Y_n^{(2)}) = 0 \\ E(X_n)^2 &= P \\ E[S_n^{(1)} Y_n^{(1)}] &= (P/2) \beta_n (1 + g|\rho_n|) \\ E[S_n^{(2)} Y_n^{(2)}] &= (P/2) \beta_n \operatorname{sgn}(\rho_n) (g + |\rho_n|) \\ \operatorname{var}(Y_n^{(1)}) &= P + \sigma^2 + \sigma_1^2 \\ \operatorname{var}(Y_n^{(2)}) &= P + \sigma^2 + \sigma_2^2 \end{aligned}$$

Note that

$$\begin{aligned} \operatorname{var}(S_n^{(1)} | Y_n^{(1)}) &= \operatorname{var}(S_n^{(1)}) - \frac{E[S_n^{(1)} Y_n^{(1)}]^2}{\operatorname{var}(Y_n^{(1)})} \\ \operatorname{var}(S_n^{(2)} | Y_n^{(2)}) &= \operatorname{var}(S_n^{(2)}) - \frac{E[S_n^{(2)} Y_n^{(2)}]^2}{\operatorname{var}(Y_n^{(2)})} \end{aligned}$$

Replacing the calculations above, we finally have

$$\begin{aligned} a_n^{(1)} &= \sqrt{\frac{\sigma^2 + \sigma_1^2 + (Pg^2(1 - \rho_n^2))/(1 + g^2 + 2g|\rho_n|)}{P + \sigma^2 + \sigma_1^2}} \\ a_n^{(2)} &= \sqrt{\frac{\sigma^2 + \sigma_2^2 + (P(1 - \rho_n^2))/(1 + g^2 + 2g|\rho_n|)}{P + \sigma^2 + \sigma_2^2}} \\ b_n^{(1)} &= \frac{(P/2) \beta_n (1 + g|\alpha_n|)}{P + \sigma^2 + \sigma_1^2} \\ b_n^{(2)} &= \frac{(P/2) \beta_n \operatorname{sgn}(\rho_n) (g + |\alpha_n|)}{P + \sigma^2 + \sigma_2^2} \end{aligned}$$

By the general time-varying coding scheme, we obtain

$$\begin{aligned} S_{n+1}^{(m)} &= \frac{1}{a_n^{(1)}} (S_n^{(1)} - b_n^{(1)} Y_n^{(1)}) \\ S_{n+2}^{(m)} &= \frac{1}{a_n^{(2)}} (S_n^{(2)} - b_n^{(2)} Y_n^{(2)}) \end{aligned}$$

Hence,

$$\begin{aligned} E[S_{n+1}^{(1)} S_{n+1}^{(2)}] &= \frac{1}{a_n^{(1)} a_n^{(2)}} \left(E[S_n^{(1)} S_n^{(2)}] - b_n^{(1)} E[S_n^{(2)} Y_n^{(1)}] \right. \\ &\quad \left. - b_n^{(2)} E[S_n^{(1)} Y_n^{(2)}] + b_n^{(1)} b_n^{(2)} E[Y_n^{(2)}] \right) \end{aligned}$$

Replacing all the results above, and perform some simple calculations we get a recursive formula for ρ_n as following

$$\rho_{n+1} = \frac{\Pi \rho_n - \frac{P\Sigma}{D_n} (g + |\rho_n|) (1 + g|\rho_n|) \operatorname{sgn}(\rho_n)}{\sqrt{\Pi} \sqrt{\left(\sigma^2 + \sigma_1^2 + \frac{Pg^2(1 - \rho_n)^2}{D_n} \right) \left(\sigma^2 + \sigma_2^2 + \frac{P(1 - \rho_n^2)}{D_n} \right)}} \quad (7)$$

where

$$\begin{aligned} \Pi &= (P + \sigma^2 + \sigma_1^2)(P + \sigma^2 + \sigma_2^2), \quad \Sigma = P + \sigma^2 + \sigma_1^2 + \sigma_2^2, \\ D_n &= 1 + g^2 + 2g|\rho_n| \end{aligned}$$

It is very difficult to affirm that the sequence $|\rho_n|$ is convergent. One strategy to overcome this situation is to keep $|\rho_n|$ unchanged (see [4]). Then we have $\rho_n = (-1)^{n+1} \rho$, where ρ is the biggest positive solution of the following equation:

$$x + \frac{\Pi x - \frac{P\Sigma}{D} (g + x) (1 + gx)}{\sqrt{\Pi} \sqrt{\left(\sigma^2 + \sigma_1^2 + \frac{Pg^2(1 - x)^2}{D} \right) \left(\sigma^2 + \sigma_2^2 + \frac{P(1 - x^2)}{D} \right)}} = 0 \quad (8)$$

where

$$\begin{aligned} \Pi &= (P + \sigma^2 + \sigma_1^2)(P + \sigma^2 + \sigma_2^2), \quad \Sigma = P + \sigma^2 + \sigma_1^2 + \sigma_2^2, \\ D &= 1 + g^2 + 2gx \end{aligned}$$

Then, we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} a_n^{(1)} &= \sqrt{\frac{\sigma^2 + \sigma_1^2 + (Pg^2(1 - \rho^2))/(1 + g^2 + 2g\rho)}{P + \sigma^2 + \sigma_1^2}} \\ \limsup_{n \rightarrow \infty} a_n^{(2)} &= \sqrt{\frac{\sigma^2 + \sigma_2^2 + (P(1 - \rho^2))/(1 + g^2 + 2g\rho)}{P + \sigma^2 + \sigma_2^2}} \end{aligned}$$

It is easy to verify that $0 < \limsup_{n \rightarrow \infty} a_n^{(m)} < 1$, $\forall m = 1, 2$. Besides, we can prove by induction that $S_n^{(m)} \sim \mathcal{N}(0, P/2)$ by using the formula

$$S_{n+1}^{(m)} = \frac{1}{a_n^{(m)}} (S_n^{(m)} - b_n^{(m)} Y_n^{(m)})$$

and replacing previously calculated $a_n^{(m)}, b_n^{(m)}$ or showing that

$$S_{n+1}^{(1)} = F_S^{-1} \circ F_{S_n^{(1)}|Y_n^{(1)}}(S_n^{(1)}|Y_n^{(1)}) \sim \mathcal{N}(0, P/2)$$

$$S_{n+1}^{(2)} = F_S^{-1} \circ F_{S_n^{(2)}|Y_n^{(2)}}(S_n^{(2)}|Y_n^{(2)}) \sim \mathcal{N}(0, P/2)$$

This means that $E[S_n^{(m)}]^2$ is a constant and upper bounded, of course. In summary, we have proved that all the constraints in the Theorem I are satisfied by using this encoding scheme. Therefore, the achievable rate region with this coding scheme is $(R_1 < R_1^*, R_2 < R_2^*)$ where

$$\begin{aligned} R_1^* &= -\limsup \log a_n^{(1)} \\ &= \frac{1}{2} \log \left(\frac{P + \sigma^2 + \sigma_1^2}{\sigma^2 + \sigma_1^2 + (Pg^2(1 - \rho^2))/D} \right) \\ R_2^* &= -\limsup \log a_n^{(2)} \\ &= \frac{1}{2} \log \left(\frac{P + \sigma^2 + \sigma_1^2}{\sigma^2 + \sigma_2^2 + (Pg^2(1 - \rho^2))/D} \right) \end{aligned}$$

where $D = 1 + g^2 + 2\rho g$ and ρ is the biggest positive solution of the equation (9). The error probabilities decay to zero as

$$\begin{aligned} -\log p_n^{(1)}(e) &= o\left(2^{2n(R_1^* - R_1)}\right) \\ -\log p_n^{(2)}(e) &= o\left(2^{2n(R_2^* - R_2)}\right) \end{aligned}$$

Remark: As shown, this encoding scheme is a variant of the Ozarow-Leung coding scheme [4] or a form of time-varying posterior matching [1]. However, the performance of this code is strictly worse than the one of LQG code [18] or Elia [7]. In the Section V below, we prove that by choosing a better choice of sequences $\alpha_n^{(m)}, b_n^{(m)}$, we can achieve better achievable rate. Specifically, our proposed coding scheme for the symmetric Gaussian broadcast channel in the Section V obtains the same achievable sum rate as LQG code [18] and strictly outperforms Kramer code [14].

VI. A CODING SCHEME FOR DEGRADED GAUSSIAN BROADCAST CHANNEL WITH FEEDBACK

For the degraded Gaussian broadcast channel, i.e., $\sigma^2 = 1, \sigma_1^2 = \sigma_2^2 = \dots = \sigma_M^2 = 0$, in which case the receivers observe the sequence. In this case, we set parameters as following:

$$P_0 = P/M$$

The modulated vector $\alpha_n = [\alpha_n^{(1)}, \alpha_n^{(2)}, \dots, \alpha_n^{(M)}]^T$ is the column $(n \bmod M + 1)$ of the Hadamard matrix M -by- M . Besides, we also set

$$a_n^{(m)} = \sqrt{\frac{\text{var}(S_n^{(m)}|Y_n^{(m)})}{P/M}}$$

$$b_n^{(m)} = \frac{E[S_n^{(m)}Y_n^{(m)}]}{\text{var}(Y_n^{(m)})}$$

where $\alpha_n = [\alpha_n^{(1)}, \alpha_n^{(2)}, \dots, \alpha_n^{(M)}]^T$ is the modulated vector above. We can show that this strategy is the same as the coding scheme used for Gaussian MAC channel with feedback proposed in [1], i.e.,

$$S_{n+1}^{(m)} = F_{S_m}^{-1} \circ F_{S_n^{(m)}|Y_n^{(m)}}(S_n^{(m)}|Y_n^{(m)})$$

with $S_m \sim \mathcal{N}(0, P/M)$.

Observe that for this physically degraded case, the Gaussian broadcast channel with feedback can be considered as M Gaussian virtual MACs with feedback, each MAC corresponds to a receiver with the inputs $S_n^{(1)}, S_n^{(2)}, \dots, S_n^{(m)}$. Apply the results of the Theorem II [1], we easily come to the following theorem.

Theorem II. Under no input power constraint, any rate vector $\{(R_1, R_2, \dots, R_M) : R_m < R_m^*\}$ is achievable for the physically degraded broadcast channel with feedback, where

$$R_m = -\limsup_{n \rightarrow \infty} \log a_n^{(m)}$$

where

$$\begin{aligned} a_n^{(m)} &= \sqrt{\frac{\text{var}(S_n^{(m)}|Y_n^{(m)})}{P/M}} \\ &= \sqrt{1 - \frac{(P/M)(\sum_{t=1}^M \rho_n^{(t,m)} \alpha_n^{(t)})^2}{(P/M)(\sum_{t=1}^M \sum_{l=1}^M \alpha_n^{(t)} \alpha_n^{(l)} \rho_n^{(t,l)}) + 1}} \end{aligned}$$

Proof: Using the same arguments as the the of the the paper [1]. \blacksquare

As a corollary [1], the degraded Gaussian broadcast channel with feedback can achieves the sum-rate

$$R_{\text{sum}} = \sum_{m=1}^M R_m^* = \frac{1}{2} \log(1 + P\lambda)$$

where λ is the biggest solution in $[1, M]$ of the following equation:

$$(Px + 1)^{M-1} = [(P/M)x(M - x) + 1]^M$$

Note that this achievable sum-rate is obtained if the input power constraint is

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n E(X_k^2) \stackrel{(a)}{=} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n P\lambda_k \stackrel{(b)}{=} P\lambda.$$

where (a) follows from the result of paper [1] and (b) follows from the Cesaro means and the fact that the eigenvalue λ_n converges to λ .

Remark: By replacing P by $P\lambda$, we see that our scheme for the degraded broadcast channel with feedback obtains the sum-rate $(1/2) \log(1 + P)$, which is the sum-rate capacity of this channel since we know that feedback does not increase the capacity of degraded Gaussian broadcast channel with feedback [10], [11], [19].

VII. A NOVEL TIME-VARYING CODING SCHEME FOR M-USER SYMMETRIC NON-DEGRADED GAUSSIAN BROADCAST CHANNEL WITH FEEDBACK

In this section, we propose a time-varying code for the non-physical degraded case, i.e., $\sigma_1^2 = \sigma_2^2 = \dots = \sigma_M^2 = 1$ and $\sigma^2 = 0$, which achieves the same performance as the LQG code [18]. Our proposed coding scheme is based on the following Lemmas:

Lemma III: Let $\lambda_1, \lambda_2, \dots, \lambda_M$ be a set of positive numbers satisfying:

$$\lambda_{m+1} = \frac{1 + (P/M)\lambda_1(M - \lambda_1)}{1 + P\lambda_1} \lambda_m, \quad m = 1, 2, \dots, M-1$$

where $\lambda_1 = \lambda$ is the biggest positive root of the following equation:

$$(Px + 1)^{M-1} = [((P/M)x(M - x) + 1)]^M \quad (9)$$

Assuming that γ is a negative number satisfying:

$$\gamma \geq -\frac{\lambda}{P\lambda + 1}$$

Then, we have $\lambda_m + \gamma > 0$ for all m .

Proof: Indeed, from the given relation, we have

$$\lambda_m = \frac{1 + P\lambda_1}{1 + (P/M)\lambda_1(M - \lambda_1)} \lambda_{m+1} > \lambda_{m+1}$$

for all $m = 1, 2, \dots, M-1$. Moreover, we have

$$\begin{aligned} \lambda_M &= \frac{\lambda_1}{1 + (P/M)\lambda_1(M - \lambda_1)} \\ &= \frac{\lambda}{1 + (P/M)\lambda(M - \lambda)} \end{aligned}$$

Combining with our assumption, then

$$\begin{aligned} \lambda_M + \gamma &\geq \lambda_M - \frac{\lambda}{P\lambda + 1} \\ &= \frac{\lambda}{1 + (P/M)\lambda(M - \lambda)} - \frac{\lambda}{P\lambda + 1} \\ &= \frac{(P/M)\lambda^3}{(1 + P\lambda)(1 + (P/M)\lambda(M - \lambda))} > 0 \end{aligned}$$

This concludes our proof. \blacksquare

Lemma IV: For any positive number λ , the following equation system has an unique solution pair (b, γ) with $b > 0, \gamma < 0$.

$$\gamma = \frac{P\lambda + 1}{1 + (P/M)\lambda(M - \lambda)} \left[\gamma + \frac{M}{P} b^2 \right] \quad (10)$$

$$\gamma = \frac{1}{4b^2} \left[Mb^2 + \frac{(P/M)\lambda^2}{1 + P\lambda} \right]^2 - \lambda \quad (11)$$

Moreover, we have

$$0 > \gamma \geq -\frac{\lambda}{1 + P\lambda}$$

and

$$Mb^2 - 2b\sqrt{\lambda + \gamma} + \frac{(P/M)\lambda^2}{1 + P\lambda} = 0$$

Proof: The equality (10) is equivalent to

$$\gamma = -\frac{(M/P)^2 b^2 (P\lambda + 1)}{\lambda^2}$$

Replacing to (11), we have

$$-\frac{(M/P)^2 b^2 (P\lambda + 1)}{\lambda^2} = \frac{1}{4b^2} \left[Mb^2 + \frac{(P/M)\lambda^2}{1 + P\lambda} \right]^2 - \lambda$$

Performing some simple manipulations, we get the quadratic equation of b^2 .

$$\begin{aligned} \left[M^2 + 4\frac{(M/P)^2 (P\lambda + 1)}{\lambda^2} \right] b^4 - 2 \left[\frac{P\lambda^2 + 2\lambda}{1 + P\lambda} \right] b^2 + \\ + \frac{(P/M)^2 \lambda^4}{(1 + P\lambda)^2} = 0 \end{aligned}$$

It is easy to see that this quadratic equation has an unique solution b^2

$$b^2 = \frac{(P\lambda^2 + 2\lambda)/(1 + P\lambda)}{M^2 + 4[(M/P)^2 (P\lambda + 1)]/\lambda^2}$$

Since $b > 0$, we get

$$b = \sqrt{\frac{(P\lambda^2 + 2\lambda)/(1 + P\lambda)}{M^2 + 4[(M/P)^2 (P\lambda + 1)]/\lambda^2}}$$

Besides, from (11) we easily see that $\gamma > -\lambda$. This means that $\lambda + \gamma > 0$. We also have

$$Mb^2 - 2b\sqrt{\lambda + \gamma} + \frac{(P/M)\lambda^2}{1 + P\lambda} = 0$$

Since this equation has a positive solution b as we mentioned, we must have

$$\lambda + \gamma \geq M \frac{(P/M)\lambda^2}{1 + P\lambda} = \frac{P\lambda^2}{1 + P\lambda}$$

or

$$\gamma \geq -\frac{\lambda}{1 + P\lambda}$$

It is obvious that $\gamma < 0$ by their relation mentioned above. That concludes our proof. \blacksquare

Theorem III: The time-varying coding scheme for M-user non-degraded Gaussian broadcast channel with feedback can achieve the following sum-rate

$$R_{sum} = \sum_{m=1}^M R_m^* = \frac{1}{2} \log(1 + P\lambda)$$

where λ is the biggest solution in $[1, M]$ of the following equation:

$$(Px + 1)^{M-1} = [(P/M)x(M - x) + 1]^M \quad (12)$$

Proof: We divide the proof into two parts. In the first part, we provide extra notations used in our analysis later. Then, in the next part, we show how to set parameters (sequences) for our encoding-decoding scheme, then prove that our coding system to achieve the same performance as the LQG code [18].

A. Extra notations:

Define a normalized covariance matrix by

$$\mathbf{R}_n = \frac{1}{(P/M)} E[\mathbf{S}_n \mathbf{S}_n^T]$$

Then

$$\begin{aligned} \mathbf{R}_n &= \frac{1}{(P/M)} \begin{bmatrix} E[S_n^{(1)} S_n^{(1)}] & \cdots & E[S_n^{(1)} S_n^{(M)}] \\ E[S_n^{(2)} S_n^{(1)}] & \cdots & E[S_n^{(2)} S_n^{(M)}] \\ \vdots & \ddots & \vdots \\ E[S_n^{(M)} S_n^{(1)}] & \cdots & E[S_n^{(M)} S_n^{(M)}] \end{bmatrix} \\ &= \begin{bmatrix} r_n^{(1,1)} & \cdots & r_n^{(1,M)} \\ r_n^{(2,1)} & \cdots & r_n^{(2,M)} \\ \vdots & \ddots & \vdots \\ r_n^{(M,1)} & \cdots & r_n^{(M,M)} \end{bmatrix} \end{aligned}$$

where

$$r_n^{(m,k)} := \frac{E[S_n^{(m)} S_n^{(k)}]}{(P/M)}$$

Now, denote by H_1, H_2, \dots, H_M the M columns of the Hadamard matrix \mathbf{H} . Set the vector $\alpha_n := [\alpha_n^{(1)}, \alpha_n^{(2)}, \dots, \alpha_n^{(M)}]^T = H_{(n \bmod M)+1}$. In addition, we also set $b_n^{(m)} = b_n \alpha_n^{(m)} \forall m$ where b_n is a real sequence. We also define an associated matrix \mathbf{G}_n as following

$$\mathbf{G}_n = \mathbf{R}_n - \gamma_n \mathbf{I}_M \quad (13)$$

where γ_n is another real sequence.

B. Parameter settings, achievable rate region analysis, and error probabilities:

Let $\lambda_1, \lambda_2, \dots, \lambda_M$ be the set of the positive numbers defined on the Lemma III. We firstly show by induction that if there exists $n_0 \in \mathbb{N}$ such that G_{n_0} is symmetric positive definite and has all of the columns of the M -by- M Hadamard matrix as its eigenvectors, then by suitably choosing sequences b_n, γ_n, β_n for all $n \geq n_0$, the matrices $\mathbf{G}_n (n \geq n_0)$ also have all these properties. In addition, by this way of parameter setting, if $\lambda_{n_0}^{(1)} = \lambda_1, \lambda_{n_0}^{(2)} = \lambda_2, \dots, \lambda_{n_0}^{(M)} = \lambda_M$, we also have $\lambda_n^{(1)} = \lambda_1, \lambda_n^{(2)} = \lambda_2, \dots, \lambda_n^{(M)} = \lambda_M$ for all $n \geq n_0$. Here, $\lambda_n^{(m)}$ is denoted by the eigenvalue associated with the eigenvector which is the column $(n \bmod M) + m$ of the M -by- M Hadamard matrix for all $m = 1, 2, \dots, M$. For brevity, denote by $\lambda_n = \lambda_n^{(1)}$, and $\lambda_1 := \lambda$, hereafter.

Indeed, the matrix \mathbf{G}_{n_0} , of course, satisfies all these properties by our assumption. Assuming that our assumption is true for some $n \geq n_0$. Denote

$$\mathbf{G}_n := \begin{bmatrix} \rho_n^{(1,1)} & \cdots & \rho_n^{(1,M)} \\ \rho_n^{(2,1)} & \cdots & \rho_n^{(2,M)} \\ \vdots & \ddots & \vdots \\ \rho_n^{(M,1)} & \cdots & \rho_n^{(M,M)} \end{bmatrix}$$

Then from (13), we obtain

$$\rho_n^{(m,k)} = r_n^{(m,k)} - \gamma_n \delta(m-k)$$

Here, we define the delta function $\delta(n) = 1$ if $n = 0$ and $\delta(n) = 0$ if $n \neq 0$.

By our encoding scheme, the transmitted signal has the following form:

$$X_n = \beta_n \sum_{t=1}^M \alpha_n^{(t)} S_n^{(t)}$$

where β_n is a coefficient to be chosen to satisfy the input power constraint. Then, the transmitted input power at time n is

$$E(X_n^2) = \beta_n^2 \frac{P}{M} \alpha_n^T \mathbf{R}_n \alpha_n = \frac{P}{M} \beta_n^2 [\alpha_n^T \mathbf{G}_n \alpha_n + \gamma_n \alpha_n^T \mathbf{I}_n \alpha_n]$$

To put it simply,

$$E(X_n^2) = \beta_n^2 \frac{P}{M} [M \lambda_n + \gamma_n M] = P \beta_n^2 (\lambda_n + \gamma_n) \quad (14)$$

(Here, we use the fact that $\mathbf{G}_n \alpha_n = \lambda_n \alpha_n$).

Moreover, the relation between the input and the output at the receiver m is expressed as

$$Y_n^{(m)} = X_n + Z_n^{(m)} = \beta_n \sum_{t=1}^M \alpha_n^{(t)} S_n^{(t)} + Z_n^{(m)}$$

Therefore,

$$\begin{aligned} E[S_n^{(m)} Y_n^{(k)}] &= E[S_n^{(m)} (\beta_n \sum_{t=1}^M \alpha_n^{(t)} S_n^{(t)} + Z_n^{(m)})] \\ &= \frac{P}{M} \beta_n \sum_{t=1}^M \alpha_n^{(t)} r_n^{(m,t)} \\ &= \frac{P}{M} \beta_n \sum_{t=1}^M \alpha_n^{(t)} [\rho_n^{(m,t)} + \gamma_n \delta(m-t)] \\ &= \frac{P}{M} \beta_n \sum_{t=1}^M \alpha_n^{(t)} \rho_n^{(m,t)} + \frac{P}{M} \beta_n \sum_{t=1}^M \alpha_n^{(t)} \gamma_n \delta(m-t) \\ &= \frac{P}{M} \beta_n \alpha_n^T \rho_n^{(m)} + \frac{P}{M} \beta_n \gamma_n \alpha_n^{(m)} \end{aligned}$$

With the assumption that \mathbf{G}_n is symmetric and that λ_n is an eigenvalue associated with the eigenvector α_n of this matrix, we have

$$\alpha_n^T \mathbf{G}_n = \alpha_n^T \mathbf{G}_n^T = \alpha_n^T \lambda_n$$

Hence, we obtain $\alpha_n^T \rho_n^{(m)} = \lambda_n \alpha_n^{(m)}$.

Combining these facts, we have

$$E[S_n^{(m)} Y_n^{(k)}] = \frac{P}{M} \beta (\lambda_n + \gamma_n) \alpha_n^{(m)} \quad (15)$$

Furthermore, we also obtain

$$\begin{aligned} E[Y_n^{(m)} Y_n^{(k)}] &= E[(X_n + Z_n^{(m)})(X_n + Z_n^{(k)})] \\ &= E(X_n^2) + E(Z_n^{(m)} Z_n^{(k)}) = P \beta_n^2 (\lambda_n + \gamma_n) + \delta(m-k) \end{aligned} \quad (16)$$

Note that by our transmission scheme

$$S_n^{(m+1)} = \frac{1}{a_n} (S_n^{(m)} - b_n \alpha_n^{(m)} Y_n^{(m)})$$

Therefore, we have

$$E[S_n^{(m+1)} S_n^{(k+1)}] = \frac{1}{a_n^2} \left(E[S_n^{(m)} S_n^{(k)}] - b_n \alpha_n^{(m)} E[S_n^{(k)} Y_n^{(m)}] \right. \\ \left. - b_n \alpha_n^{(k)} E[S_n^{(m)} Y_n^{(k)}] + b_n^2 \alpha_n^{(m)} \alpha_n^{(k)} E[Y_n^{(m)} Y_n^{(k)}] \right)$$

Then,

$$\frac{P}{M} r_{n+1}^{(m,k)} = \frac{1}{a_n^2} \left(\frac{P}{M} r_n^{(m,k)} - b_n \alpha_n^{(m)} \frac{P}{M} \beta_n (\lambda_n + \gamma_n) \alpha_n^{(k)} \right. \\ \left. - b_n \alpha_n^{(k)} \frac{P}{M} \beta_n (\lambda_n + \gamma_n) \alpha_n^{(m)} \right. \\ \left. + b_n^2 \alpha_n^{(m)} \alpha_n^{(k)} [P \beta_n^2 (\lambda_n + \gamma_n) + \delta(m-k)] \right) \\ = \frac{1}{a_n^2} \left(\frac{P}{M} r_n^{(m,k)} - 2b_n \beta_n \frac{P}{M} (\lambda_n + \gamma_n) \alpha_n^{(m)} \alpha_n^{(k)} \right. \\ \left. + b_n^2 \alpha_n^{(m)} \alpha_n^{(k)} [P \beta_n^2 (\lambda_n + \gamma_n) + \delta(m-k)] \right)$$

Therefore, we have

$$\rho_{n+1}^{(m,k)} + \gamma_{n+1} \delta(m-k) = \frac{1}{a_n^2} \left(\rho_n^{(m,k)} + \gamma_n \delta(m-k) \right. \\ \left. - 2b_n \beta_n (\lambda_n + \gamma_n) \alpha_n^{(m)} \alpha_n^{(k)} + M b_n^2 \beta_n^2 (\lambda_n + \gamma_n) \alpha_n^{(m)} \alpha_n^{(k)} \right. \\ \left. + \frac{M}{P} b_n^2 \alpha_n^{(m)} \alpha_n^{(k)} \delta(m-k) \right) \quad (17)$$

Now, if we set

$$a_n = \sqrt{\frac{1 + (P/M) \lambda_n (M - \lambda_n)}{1 + P \lambda_n}} \quad (18)$$

and choose

$$\gamma_{n+1} = \frac{1}{a_n^2} \left(\gamma_n + \frac{M}{P} b_n^2 \right) \quad (19)$$

Then for all m, k we have

$$\gamma_{n+1} \delta(m-k) = \frac{1}{a_n^2} \left(\gamma_n \delta(m-k) + \frac{M}{P} b_n^2 \alpha_n^{(m)} \alpha_n^{(k)} \delta(m-k) \right)$$

Combining with (17), we obtain

$$\rho_{n+1}^{(m,k)} = \frac{1}{a_n^2} \left[\rho_n^{(m,k)} - 2b_n \beta_n (\lambda_n + \gamma_n) \alpha_n^{(m)} \alpha_n^{(k)} \right. \\ \left. + M b_n^2 \beta_n^2 (\lambda_n + \gamma_n) \alpha_n^{(m)} \alpha_n^{(k)} \right] \\ = \frac{1}{a_n^2} \left(\rho_n^{(m,k)} - [2b_n \beta_n (\lambda_n + \gamma) - M b_n^2 \beta_n^2 (\lambda_n + \gamma_n)] \alpha_n^{(m)} \alpha_n^{(k)} \right) \quad (20)$$

Now, for all $n \geq n_0$, we will set $b_n = b > 0, \gamma_n = \gamma < 0$ which is the unique solution in the equation system in the Lemma IV above. With the induction assumption that $\lambda_n = \lambda$, in order for satisfying the input power constraint, we can set

$$\beta_n = \frac{1}{\sqrt{\lambda_n + \gamma_n}} = \frac{1}{\sqrt{\lambda + \gamma}}$$

Note that this definition of β_n is well-defined for the Lemma IV $\lambda + \gamma > 0$. Moreover, from (14), we see that $E(X_n^2) = P$

for all $n \geq n_0$.

In addition, we also see that

$$2b_n \beta_n (\lambda_n + \gamma_n) - M b_n^2 \beta_n^2 (\lambda_n + \gamma_n) \\ = 2b \sqrt{\lambda + \gamma} - M b^2 = \frac{(P/M) \lambda^2}{1 + P \lambda} = \frac{(P/M) \lambda_n^2}{1 + P \lambda_n} \quad (21)$$

Combining these facts, from (21) come to the following recursion

$$\rho_{n+1}^{(m,k)} = \frac{1 + P \lambda_n}{1 + (P/M) \lambda_n (M - \lambda_n)} \rho_n^{(m,k)} \\ - \frac{(P/M) \lambda_n^2}{1 + (P/M) \lambda_n (M - \lambda_n)} \alpha_n^{(m)} \alpha_n^{(k)} \quad (22)$$

This leads to

$$\mathbf{G}_{n+1} = \frac{1 + P \lambda_n}{1 + (P/M) \lambda_n (M - \lambda_n)} \mathbf{G}_n \\ - \frac{(P/M) \lambda_n^2}{1 + (P/M) \lambda_n (M - \lambda_n)} \alpha_n \alpha_n^T$$

Since we assume that the matrix \mathbf{G}_n is symmetric, the matrix \mathbf{G}_{n+1} is obviously symmetric. Denote $\mathbf{H}_n = [\alpha_n \ \alpha_{n+1} \ \dots \ \alpha_{n+M-1}]$ by our induction assumption. We see that the columns of \mathbf{H}_n creates M linearly independent eigenvectors of the matrix \mathbf{R}_n . Furthermore,

$$\mathbf{H}_{n+1}^T \mathbf{G}_{n+1} \mathbf{H}_{n+1} = \frac{1 + P \lambda_n}{1 + (P/M) \lambda_n (M - \lambda_n)} \mathbf{H}_{n+1}^T \mathbf{G}_n \mathbf{H}_{n+1} \\ - \frac{(P/M) \lambda_n^2}{1 + (P/M) \lambda_n (M - \lambda_n)} \mathbf{H}_{n+1}^T \alpha_n \alpha_n^T \mathbf{H}_{n+1} \quad (23)$$

Note that since all columns of \mathbf{H}_n are eigenvectors of the matrix \mathbf{G}_n , so all the columns of the matrix \mathbf{H}_{n+1} are also eigenvectors of the matrix \mathbf{R}_n , hence we has the following eigenvalue decomposition

$$\mathbf{\Lambda}_n = \frac{1}{M} \mathbf{H}_{n+1}^T \mathbf{G}_n \mathbf{H}_{n+1} \quad (24)$$

$$\mathbf{\Lambda}_{n+1} = \frac{1}{M} \mathbf{H}_{n+1}^T \mathbf{G}_{n+1} \mathbf{H}_{n+1}$$

where $\mathbf{\Lambda}_n = \text{diag}(\lambda_n^{(2)}, \lambda_n^{(3)}, \dots, \lambda_n^{(M)}, \lambda_n^{(1)}); \mathbf{\Lambda}_{n+1} = \text{diag}(\lambda_{n+1}^{(1)}, \lambda_{n+1}^{(2)}, \dots, \lambda_{n+1}^{(M)})$, which are diagonal matrices.

We also have

$$\mathbf{H}_{n+1}^T \alpha_n \alpha_n^T \mathbf{H}_{n+1} = [\alpha_n^T \mathbf{H}_{n+1}]^T \alpha_n^T \mathbf{H}_{n+1} \quad (25)$$

where

$$\alpha_n^T \mathbf{H}_{n+1} = \alpha_n^T [\alpha_{n+1} \ \alpha_{n+2} \ \dots \ \alpha_{n+M}] \\ = [0 \ 0 \ \dots \ M] \quad (26)$$

From (23), (24), (25), (26), the matrix $\mathbf{H}_{n+1}^T \mathbf{G}_{n+1} \mathbf{H}_{n+1}$ must be a diagonal one since the right side of (23) is a diagonal matrix. Hence, all columns of the matrix \mathbf{H}_{n+1} are eigenvectors of the matrix \mathbf{R}_{n+1} .

Moreover, from (23) and our notations mentioned above, we also have

$$\lambda_{n+1}^{(k)} = \begin{cases} \frac{(1+P\lambda_n^{(1)})\lambda_n^{(k+1)}}{(1+(P/M)\lambda_n^{(1)}(M-\lambda_n^{(1)}))}, & k = 1, 2, \dots, M-1 \\ \frac{\lambda_n^{(1)}}{1+(P/M)\lambda_n^{(1)}(M-\lambda_n^{(1)})}, & k = M \end{cases} \quad (27)$$

for all $k \in \{1, 2, \dots, M\}$.

Since we assumed that all eigenvalues of \mathbf{G}_n are positive (\mathbf{G}_n symmetric positive definite), from (27) we see that all eigenvalues of \mathbf{G}_{n+1} are also positive. Note that we also confirmed that \mathbf{G}_{n+1} is symmetric above, therefore \mathbf{G}_{n+1} is a symmetric positive definite matrix. In short, if \mathbf{G}_n is a symmetric positive definite matrix and has all columns of the Hadamard matrix as its eigenvectors, then \mathbf{G}_{n+1} has all these properties. Moreover, by our induction assumption that $\lambda_n^{(m)} = \lambda_m$, $\forall m$ and the relation among λ_m 's defined on the Lemma III, from the recursion formula (28) we easily come to conclusion that $\lambda_{n+1}^{(m)} = \lambda_m$ for all $m = 1, 2, \dots, M$. That concludes our proof by induction.

Now, assuming that

$$\mathbf{G}_1 = \lambda_0 \mathbf{I}_M \quad (28)$$

for some $\lambda_0 > 0$ (The value of λ_0 will be defined later). We will show that by judiciously choosing of parameters (sequences) $\gamma_n, b_n, \beta_n, a_n, b_n$ at the first M transmissions, i.e., $1 \leq n \leq M$, we can force the matrix \mathbf{G}_M to be symmetric positive definite matrix and have all the columns of the Hadamard matrix as its eigenvectors. In addition, we can achieve $b_M = b > 0, \gamma_M = \gamma < 0$ which is the unique solution of the equation system in the Lemma IV. Our argument is as following:

For $1 \leq n \leq M-1$, we set parameters as following:

$$a_n = \sqrt{\frac{1 + (P/M)\lambda(M-\lambda)}{1 + P\lambda}} := a \quad (29)$$

where λ is the biggest positive root of the equation (9), Lemma III. We also set

$$b_n = b, \gamma_n = \gamma; 1 \leq n \leq M-1$$

where $b > 0, \gamma < 0$ is the unique solution of the equation system in the Lemma IV. We will show by induction that a judiciously choosing of the sequence $\beta_n, 1 \leq n \leq M-1$ can be used to force all the matrices $\mathbf{G}_n, 1 \leq n \leq M$ to be symmetric positive definite and have all the M -by- M Hadamard matrix columns as their eigenvectors.

Indeed, the matrix $\mathbf{G}_1 = \lambda_0 \mathbf{I}_M$, of course, satisfies these properties. Now assume that the matrix \mathbf{G}_n has these properties for some $n \geq 1$, then we can still define $\lambda_n^{(m)}$ as the eigenvalue associated with eigenvector which is the column $(n \bmod M) + 1$ of the M -by- M Hadamard matrix as above for all $1 \leq m \leq M$. With this assumption, all the arguments above are unchanged. Specifically, we have

$$\gamma_{n+1} = \frac{1}{a_n^2} \left(\gamma_n + \frac{M}{P} b_n^2 \right) \quad (30)$$

(since we already fixed $\gamma_n = \gamma, b_n = b, a_n = a$ for $1 \leq n \leq M-1$ as mentioned above together with the results of Lemma IV).

Moreover, if we can find a positive sequence $d_n, 1 \leq n \leq M-1$ such that the quadratic equation

$$2b_n \beta_n (\lambda_n + \gamma_n) - M b_n^2 \beta_n^2 (\lambda_n + \gamma_n) = \left(\frac{1-d_n}{M} \right) \lambda_n \quad (31)$$

has solution β_n , then from (20) for $1 \leq n \leq M-1$, we have

$$\rho_{n+1}^{(k)} = \frac{1}{a^2} \left[\rho_n^{(m,k)} - \frac{(1-d_n)}{M} \lambda_n \alpha_n^{(m)} \alpha_n^{(k)} \right]$$

It follows that

$$\mathbf{G}_{n+1} = \frac{1}{a^2} \left[\mathbf{G}_n - \frac{(1-d_n)}{M} \lambda_n \alpha_n \alpha_n^T \right] \quad (32)$$

Hence, the equation (27) becomes to

$$\lambda_{n+1}^{(k)} = \begin{cases} (1/a^2) \lambda_n^{(k+1)}, & k = 1, 2, \dots, M-1 \\ (d_n/a^2) \lambda_n^{(1)}, & k = M \end{cases} \quad (33)$$

for all $k \in \{1, 2, \dots, M\}$.

From (32) and (33), it is easy to see that if the matrix \mathbf{G}_n is symmetric positive definite for some $n, 1 \leq n \leq M-1$, then the matrix \mathbf{G}_{n+1} also has these properties.

To complete our proof for this part, we will show how to choose the sequence $d_n, 1 \leq n \leq M-1$ such that the equation (31) has solution β_n . More importantly, we show how to choose λ_0 to force the matrix \mathbf{G}_M has the fixed eigenvalue set $\{\lambda_1, \lambda_2, \dots, \lambda_M\}$ defined in the Lemma III. Our procedure is introduced below.

Applying (33) $M-1$ times, for $1 \leq m \leq M$ we obtain

$$\begin{aligned} \lambda_M^{(m)} &= \frac{d_{M-1}}{a^{2(M-1)}} \lambda_0 = \left[\frac{1 + P\lambda}{1 + (P/M)\lambda(M-\lambda)} \right]^{M-1} d_{M-1} \lambda_0 \\ &= \left[\frac{[1 + P\lambda]^{M-1}}{[1 + (P/M)\lambda(M-\lambda)]^M} \right] [1 + (P/M)\lambda(M-\lambda)] d_{M-1} \lambda_0 \end{aligned}$$

where $d_0 := 1$. In summary, we have

$$\lambda_M^{(m)} = [1 + (P/M)\lambda(M-\lambda)] d_{m-1} \lambda_0$$

To force other eigenvalues $\lambda_M^{(m)} = \lambda_m$, we must solve the following equation to find d_{m-1}

$$[1 + (P/M)\lambda(M-\lambda)] d_{m-1} \lambda_0 = \lambda_m \quad (34)$$

From (34), it is easy to see that $\lambda_M^{(1)} = \lambda_1 = \lambda$ if we set

$$\lambda_0 = \frac{\lambda}{1 + (P/M)\lambda(M-\lambda)} \quad (35)$$

Replacing the chosen value of λ_0 , we must have

$$d_{m-1} = \frac{\lambda_m}{\lambda}$$

Besides, from the Lemma III, we can easily show that

$$\lambda_m = a^{2(m-1)} \lambda, \quad 1 \leq m \leq M$$

Combining these facts, finally we must choose

$$d_{m-1} = a^{2(m-1)},$$

for all $1 \leq m \leq M$. This means that

$$d_n = a^{2n}, \forall 1 \leq n \leq M-1 \quad (36)$$

Now, to complete the induction proof for this part, we need to show that the equation (31) has a positive solution β_n for the choice of the positive sequence d_n in (36). Observe that with chosen values, i. e., $\beta_n = \beta, \gamma_n = \gamma$, (31) is a quadratic equation in $\beta_n b$ for each $1 \leq n \leq M-1$. This quadratic equation has a solution $\beta_n b$ if

$$(\lambda_n + \gamma)^2 \geq (\lambda_n + \gamma)M \frac{1 - d_n}{M} \lambda_n$$

or

$$(\lambda_n + \gamma)(\gamma + d_n \lambda_n) \geq 0 \quad (37)$$

Note that from equations (33), (35) we can easily see that for $1 \leq n \leq M$ then

$$\begin{aligned} \lambda_n &= \lambda_n^{(1)} = \frac{1}{a^{2(n-1)}} \lambda_0 \\ &= \frac{1}{a^{2(n-1)}} \left[\frac{\lambda}{1 + (P/M)\lambda(M-\lambda)} \right] \end{aligned}$$

Therefore, combining with (36), for $1 \leq n \leq M-1$, we obtain

$$\begin{aligned} \gamma + d_n \lambda_n &= \gamma + a^{2n} \frac{1}{a^{2(n-1)}} \left[\frac{\lambda}{1 + (P/M)\lambda(M-\lambda)} \right] \\ &= \gamma + a^2 \left[\frac{\lambda}{1 + (P/M)\lambda(M-\lambda)} \right] \\ &= \gamma + \frac{1 + (P/M)\lambda(M-\lambda)}{1 + P\lambda} \left[\frac{\lambda}{1 + (P/M)\lambda(M-\lambda)} \right] \\ &= \gamma + \frac{\lambda}{1 + P\lambda} \geq 0 \end{aligned}$$

Here, the last equality follows from the Lemma IV. On the other hand, since

$$a^2 = \frac{1 + (P/M)\lambda(M-\lambda)}{1 + P\lambda} < 1$$

then $d_n = a^{2n} < 1$. Therefore,

$$\gamma + \lambda_n \geq \gamma + d_n \lambda_n \geq 0$$

From these facts, the inequality (37) is, of course, satisfied. It is easy to show that the equation (31) indeed has two positive solutions with that choice of the sequence d_n . We can choose any solution of these as the value of $\beta_n b$, but in order for reducing the transmitted power in the first M transmission, we can choose $\beta_n b$ is the smallest one. To finish the induction proof, we need to show that our encoding scheme is realizable by proving that \mathbf{R}_1 is a diagonal matrix the same elements in its main diagonal. This can be showed by observing that

$$\begin{aligned} \mathbf{R}_1 &= \mathbf{G}_1 + \gamma \mathbf{I}_M = (\lambda_0 + \gamma) \mathbf{I}_M \\ &= \left[\frac{\lambda}{1 + (P/M)\lambda(M-\lambda)} + \gamma \right] \mathbf{I}_M \end{aligned}$$

By the Lemma IV, then

$$\gamma \geq -\frac{\lambda}{P\lambda + 1}$$

Hence,

$$\frac{\lambda}{1 + (P/M)\lambda(M-\lambda)} + \gamma > \frac{\lambda}{1 + P\lambda} + \gamma \geq 0$$

This means that we can set the initialized random variable (see our encoding scheme in the Section III)

$$S \sim \mathcal{N} \left(0, (P/M) \left[\frac{\lambda}{1 + (P/M)\lambda(M-\lambda)} + \gamma \right] \right)$$

or

$$P_0 = \frac{P}{M} \left[\frac{\lambda}{1 + (P/M)\lambda(M-\lambda)} + \gamma \right] > 0$$

Finally, we find the achievable rate vector and error probabilities. By our encoding scheme, we know that

$$E(X_n^2) = P \quad \forall n \geq M$$

Then, by Casero mean Lemma, we easily to see that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n E(X_k^2) \rightarrow P$$

This means that the input power constraint is satisfied with this encoding scheme. Besides, for all $n \geq M$, we also have

$$\begin{aligned} \sum_{m=1}^M W_n^{(m)} &:= \sum_{m=1}^M E[S_n^{(m)}]^2 = \frac{P}{M} \text{tr}(\mathbf{R}_n) \\ &= \frac{P}{M} (\text{tr}(\mathbf{G}_n) + M\gamma) = \frac{P}{M} \left(\sum_{m=1}^M \lambda_m + M\gamma \right) < \infty \end{aligned}$$

Therefore, we have $W_n := \sup_m W_n^{(m)} < \infty$ since M is finite.

Furthermore, since $1 \leq \lambda \leq M$, we have

$$0 < \limsup_{n \rightarrow \infty} a_n = \frac{1 + (P/M)\lambda(M-\lambda)}{P\lambda + 1} < 1$$

In summary, the two constraints in the Theorem I are both satisfied. Applying the result of this theorem, we have any rate less than

$$\begin{aligned} R_m^* &= -\limsup \log a_n \\ &= \frac{1}{2} \liminf_{n \rightarrow \infty} \log \left(\frac{1 + P\lambda_n}{1 + (P/M)\lambda_n(M-\lambda_n)} \right) \\ &= \frac{1}{2} \log \left(\frac{1 + P\lambda}{1 + (P/M)\lambda(M-\lambda)} \right) \end{aligned}$$

is achievable for the connection between the transmitter and the receiver m , for all $m = 1, 2, \dots, M$. Hence, any sum rate which is less than

$$\begin{aligned} \sum_{m=1}^M R_m^* &= \frac{M}{2} \log \left(\frac{1 + P\lambda}{1 + (P/M)\lambda(M-\lambda)} \right) \\ &= \frac{1}{2} \log \left(\left[\frac{1 + P\lambda}{1 + (P/M)\lambda(M-\lambda)} \right]^M \right) = \frac{1}{2} \log(1 + P\lambda) \end{aligned}$$

is achievable, where λ is the biggest solution in the $[1, M]$ of the equation (9). This result coincides with the Theorem

2 [18]. With this result, like MAC case, we can prove that for large M , we have

$$\sum_{m=1}^M R_m \approx \frac{1}{M} \log M + \frac{1}{2} \log \log M$$

(refer the formula (72), [16]). This means the difference of the sum rate of broadcast channel as well as MAC channels compared with no-feedback case grows as $(\log \log M)/2$.

C. Comparison with other well-known AWGN-BC

The time-varying coding scheme approach has been applied to the AWGN-MAC with feedback. It is shown in [1] that the time-varying coding scheme can achieve the linear feedback sum capacity [16] as the Kramer code [14] or LQG code [17]. Let $R_{MAC}(M, P)$ denote the symmetric sum-rate achievable by the time-varying code [1] for M -user AWGN MAC with feedback where each sender has power constraint P . Then, we have [1, Theorem III],

$$R_{MAC}(M, P) = \frac{1}{2} \log(1 + MP\lambda)$$

where λ is the biggest solution to

$$(1 + MPx)^{M-1} = (1 + Px(M - x))^M$$

Comparing with the Theorem III, it is easy to see that

$$R_{BC}(M, P) = R_{MAC}(M, P/M)$$

This shows that under the same sum-power constraint P , the sum rate achievable by the time-varying code over MAC and BC is equal. This connection between MAC and the BC is already mentioned in [18]. Our result confirms the duality between broadcast and MAC channel. ■

VIII. CONCLUSION

A general coding strategy for the Gaussian broadcast channel with feedback was proposed, and achievable rate region, error performance were drawn. Then, based on the error analysis of performance of this coding strategy, we designed a novel coding scheme which obtain the same sum-rate as the LQG code [18]. Besides, we show that a variant of Ozarow-Leung's coding scheme can be obtained by using this strategy. We also proposed a coding scheme for the degraded Gaussian broadcast channel. An interesting further research topic is to find a good sequences $a_n^{(m)}, b_n^{(m)}$ to obtain good asymptotic capacity for more general settings in [19], [20].

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